Vector differential operators in curvilinear coordinates

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Abstract:
These are some notes on vector differential operators in curvilinear coordinates. In particular, we derive the expressions for gradient, divergence, curl, and Laplacian in a general curvilinear coordinate system.

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1 Introduction

Different laws of physics are constructed by using different differential operators. While most common is the Cartesian coordinate system, depending on the symmetry of the problem, other coordinate systems are amenable for different calculations. One particular example is the system possessing a rotational symmetry where spherical coordinate system is clearly the best suited for most of the calculations. The other example is any system possessing an axial symmetry where cylindrical coordinate system is clearly preferred.
Vector differential operators enjoys its ubiquitous appearances in various branches of physics, notably in Electrodynamics and Quantum Mechanics. They are usually presented in Cartesian coordinates as gradient, divergence, curl, and Laplacian with the following expressions

$$\vec{\nabla}\Phi(x,y,z) = \hat{e}_x\frac{\partial \Phi}{\partial x} + \hat{e}_y\frac{\partial \Phi}{\partial y} + \hat{e}_z\frac{\partial \Phi}{\partial z}, \tag{1}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A^x}{\partial x} + \frac{\partial A^y}{\partial y} + \frac{\partial A^z}{\partial z}, \tag{2}$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A^x & A^y & A^z \end{vmatrix}, \tag{3}$$

$$\nabla^2\Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}, \tag{4}$$

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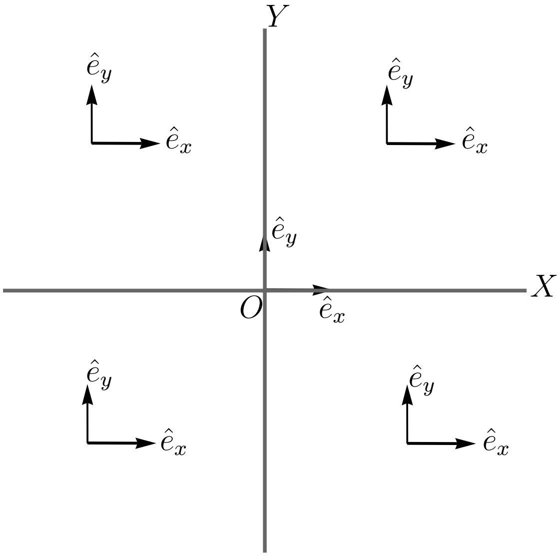
where the grad and the Laplacian operators act on a scalar $\Phi(x, y, z)$, the divergence and curl operator acts on a vector[2]

$$\vec{A} = A^x \hat{e}_x + A^y \hat{e}_y + A^z \hat{e}_z = \sum_{i=1}^{3} A^i \hat{e}_i. \tag{5}$$

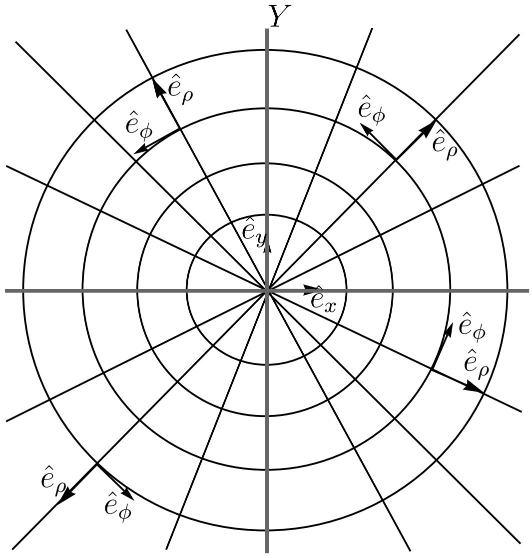
It will be useful to have expressions of gradient, divergence, curl, and Laplacian in other coordinate systems. This is the main purpose on these notes.[3]

2 Curvilinear coordinates

In this section, we will introduce the curvilinear coordinates. While Cartesian coordinates uses perpendicular lines as the axes, other coordinate lines can be useful. Particularly useful other coordinates are: 1) Polar coordinates, 2) Spherical coordinates, 3) Cylindrical coordinates. In all these coordinate systems, the coordinate axes are curved. They belong to curvilinear coordinate systems.
One important difference between the Cartesian and curvilinear coordinates is that in Cartesian coordinates the unit vectors are fixed, whereas in the curvilinear coordinates the unit vectors are not fixed. This is illustrated in the following figures. We consider three curvilinear coordinates $\{q^1, q^2, q^3\}$. Their dependence on the Cartesian coordinates are given by the following relations



(a) For Cartesian coordinates the coordinate unit vectors $\{\hat{e}_x, \hat{e}_y\}$ are fixed irrespective to the base point.



(b) In polar coordinates (ρ, ϕ), the unit vectors $\{\hat{e}_\rho, \hat{e}_\phi\}$ depend on the position. This is in contrast to the Cartesian coordinates.

$$q^1 = q^1(x, y, z), \tag{6}$$
$$q^2 = q^2(x, y, z), \tag{7}$$
$$q^3 = q^3(x, y, z). \tag{8}$$

These relations can be inverted, and we can write

$$x = x(q^1, q^2, q^3), \tag{9}$$
$$y = y(q^1, q^2, q^3), \tag{10}$$
$$z = z(q^1, q^2, q^3). \tag{11}$$

[2]In our notation $A^1 = A^x,\ A^2 = A^y,\ A^3 = A^z$, and $\hat{e}_1 = \hat{e}_x,\ \hat{e}_2 = \hat{e}_y,\ \hat{e}_3 = \hat{e}_z$.
[3]These notes are written from the author's understanding, and not original. Significant help have been taken from [1, 2, 3, 4, 5, 6].

In the usual vector analysis we write the position vector

$$\vec{\ell} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z. \tag{12}$$

The dependence on $\{q^i\}$[4] can be obtained by inserting Eqs. (9)-(11) into Eq. (12), and we have $\vec{\ell} = \vec{\ell}(q^1, q^2, q^3)$.

The quantity $\frac{\partial \vec{\ell}}{\partial q^i}$ denotes a vector in the direction of increasing q^i while keeping other coordinates fixed. Therefore we can write

$$\vec{q}_i = \frac{\partial \vec{\ell}}{\partial q^i}. \tag{13}$$

Therefore the unit vector in the direction of q^i is given by

$$\hat{q}_i = \frac{\vec{q}_i}{|\vec{q}_i|}. \tag{14}$$

2.1 The metric

The central quantity from which the expressions of vector differential operators will be obtained is the metric. The metric can be extracted from the line element which is the distance squared of two infinitesimally close points. In Cartesian coordinates, the distance squared between two pints $P(x, y, z)$ and $Q(x + dx, y + dy, z + dz)$ can be obtained from Pythagoras theorem

$$ds^2 = dx^2 + dy^2 + dz^2 = d\vec{\ell} \cdot d\vec{\ell}. \tag{15}$$

In general we can write

$$d\vec{\ell} = \sum_{i=1}^{3} \frac{\partial \vec{\ell}}{\partial q^i} dq^i = \sum_{i=1}^{3} \vec{q}_i dq^i. \tag{16}$$

Inserting this into Eq. (15) we can write

$$ds^2 = \sum_{i,j=1}^{3} \vec{q}_i \cdot \vec{q}_j dq^i dq^j, \tag{17}$$

$$= \sum_{i,j=1}^{3} g_{ij} dq^i dq^j. \tag{18}$$

The quantity $g_{ij} = \vec{q}_i \cdot \vec{q}_j$ is called the metric, and is a central quantity in differential geometry. Although not necessary, we will assume the the coordinate vectors are orthogonal, that means

$$\vec{q}_i \cdot \vec{q}_j = \begin{cases} \text{non-zero if } i = j, \\ 0 \text{ if } i \neq j. \end{cases} \tag{19}$$

In this case, g_{ij} is a diagonal matrix

$$g_{ij} = \begin{pmatrix} (h_1)^2 & 0 & 0 \\ 0 & (h_2)^2 & 0 \\ 0 & 0 & (h_3)^2 \end{pmatrix} \tag{20}$$

where $h_1 = |\vec{q}_1|$, $h_2 = |\vec{q}_2|$, $h_3 = |\vec{q}_3|$. Then the line element for an orthogonal curvilinear coordinates can be written as

$$ds^2 = (h_1)^2(dq^1)^2 + (h_2)^2(dq^2)^2 + (h_3)^2(dq^3)^2. \tag{21}$$

Using the information above we can also write

$$d\vec{\ell} = h_1 dq^1 \hat{q}_1 + h_2 dq^2 \hat{q}_2 + h_3 dq^3 \hat{q}_3. \tag{22}$$

Eqs. (21)-(22) will be elemental in the following discussions.

[4] Small Latin indices can take values $i, j, k = 1, 2, 3$.

3 Vector differential operators

In this section, we will construct different vector differential operators. We will start from the gradient operator which will be required to construct the other differential operators.

3.1 Gradient

The gradient of a scalar functions measures the steepness of a function along a given direction. More precisely, consider a function $\Phi(x, y, z)$. Its change is given by:

$$d\Phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz, \tag{23}$$

$$= (\vec{\nabla}\Phi) \cdot \vec{d\ell}. \tag{24}$$

Since $\vec{d\ell} = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z$, we can extract the form of the gradient operator in Cartesian coordinates as

$$\vec{\nabla} = \hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y} + \hat{e}_z\frac{\partial}{\partial z}. \tag{25}$$

The same reasoning can be applied to find the form of the gradient in any curvilinear coordinate system. Consider a scalar function $\Phi(q^1, q^2, q^3)$. The differential of this function can be computed as follows:

$$d\Phi = \frac{\partial \Phi}{\partial q^1}dq^1 + \frac{\partial \Phi}{\partial q^2}dq^2 + \frac{\partial \Phi}{\partial q^3}dq^3, \tag{26}$$

$$= (\vec{\nabla}\Phi) \cdot \vec{d\ell}. \tag{27}$$

We know from Eq. (22) that

$$\vec{d\ell} = h_1 dq^1 \hat{q}_1 + h_2 dq^2 \hat{q}_2 + h_3 dq^3 \hat{q}_3. \tag{28}$$

Therefore, Eq. (27) will be satisfied when

$$\vec{\nabla} = \frac{\hat{q}_1}{h_1}\frac{\partial}{\partial q^1} + \frac{\hat{q}_2}{h_2}\frac{\partial}{\partial q^2} + \frac{\hat{q}_3}{h_3}\frac{\partial}{\partial q^3}. \tag{29}$$

This expression will also be abbreviated as

$$\vec{\nabla} = \sum_{i=1}^{3} \frac{\hat{q}_i}{h_i}\partial_i \tag{30}$$

where

$$\partial_i = \frac{\partial}{\partial q^i} \tag{31}$$

has been defined for brevity.

3.2 Divergence

Having obtained the expression for gradient, now we will move to obtaining an expression for divergence of a vector. Consider a vector expressed in the basis $\{\hat{q}_i\}$. That means

$$\vec{A} = A^1\hat{q}_1 + A^2\hat{q}_2 + A^3\hat{q}_3 = \sum_{i=1}^{3} A^i\hat{q}_i = \sum_{i=1}^{3} \frac{A^i}{h_i}\vec{q}_i. \tag{32}$$

To take the divergence, we perform the following steps

$$\vec{\nabla} \cdot \vec{A} = \sum_{i=1}^{3} \frac{\hat{q}_i}{h_i} \partial_i \cdot \left(\sum_{j=1}^{3} \frac{A^j}{h_j} \vec{q}_j \right), \tag{33}$$

$$= \sum_{i,j=1}^{3} \left[\frac{h_j \hat{q}_i \cdot \hat{q}_j}{h_i} \partial_i \left(\frac{A^j}{h_j} \right) + \frac{A^j}{h_i h_j} \hat{q}_i \cdot \partial_i \vec{q}_j \right], \tag{34}$$

$$= \sum_{i=1}^{3} \partial_i \left(\frac{A^i}{h_i} \right) + \sum_{i,j,k=1}^{3} \frac{A^j}{h_i h_j} \Gamma^k_{ij} \hat{q}_i \cdot \vec{q}_k, \tag{35}$$

$$= \sum_{i=1}^{3} \partial_i \left(\frac{A^i}{h_i} \right) + \sum_{i,j,k=1}^{3} \frac{A^j h_k}{h_i h_j} \Gamma^k_{ij} \hat{q}_i \cdot \hat{q}_k, \tag{36}$$

$$= \sum_{i=1}^{3} \partial_i \left(\frac{A^i}{h_i} \right) + \sum_{i,j=1}^{3} \frac{A^j}{h_j} \Gamma^i_{ij}. \tag{37}$$

To simplify the above, we have used the orthonormality property of the unit vectors, namely $\hat{q}_i \cdot \hat{q}_j = \delta_{ij}$. We have also used the fact that

$$\partial_i \vec{q}_j = \sum_{k=1}^{3} \Gamma^k_{ij} \hat{q}_k. \tag{38}$$

This is a simple consequence of the fact that the set $\{\hat{q}_i\}$ forms a basis set at the particular point.
To find an expression for the coefficients Γ^k_{ij}, we perform the following manipulations

$$\partial_i g_{jk} = \partial_i \left(\vec{q}_j \cdot \vec{q}_k \right), \tag{39}$$

$$= (\partial_i \vec{q}_j) \cdot \vec{q}_k + \vec{q}_j \cdot (\partial_i \vec{q}_k), \tag{40}$$

$$= \sum_{m=1}^{3} \Gamma^m_{ij} \vec{q}_m \cdot \vec{q}_k + \sum_{m=1}^{3} \Gamma^m_{ik} \vec{q}_j \cdot \vec{q}_m, \tag{41}$$

$$= \sum_{m=1}^{3} \Gamma^m_{ij} g_{mk} + \sum_{m=1}^{3} \Gamma^m_{ik} g_{jm}. \tag{42}$$

Before moving further, we will need one important property of Γ^k_{ij}. This can be found from the following manipulations. First observe that

$$\partial_i \vec{q}_j = \partial_i \partial_j \vec{\ell} = \partial_j \partial_i \vec{\ell} = \partial_j \vec{q}_i. \tag{43}$$

This implies

$$\partial_i \vec{q}_j = \partial_j \vec{q}_i, \tag{44}$$

$$\Rightarrow \sum_{k=1}^{3} \Gamma^k_{ij} \vec{q}_k = \sum_{k=1}^{3} \Gamma^k_{ji} \vec{q}_k, \tag{45}$$

$$\Rightarrow \Gamma^k_{ij} = \Gamma^k_{ji}. \tag{46}$$

That means Γ^k_{ij} is symmetric in its lower indices.
From Eq. (42) we can write

$$\partial_i g_{jm} = \sum_{n=1}^{3} \Gamma^n_{ij} g_{nm} + \sum_{n=1}^{3} \Gamma^n_{im} g_{jn}, \tag{47}$$

$$\partial_j g_{mi} = \sum_{n=1}^{3} \Gamma^n_{jm} g_{ni} + \sum_{n=1}^{3} \Gamma^n_{ji} g_{mn}, \tag{48}$$

$$\partial_m g_{ij} = \sum_{n=1}^{3} \Gamma^n_{mi} g_{nj} + \sum_{n=1}^{3} \Gamma^n_{mj} g_{in}. \tag{49}$$

Now performing (47)+(48)-(49), and using the symmetry of Γ^n_{ij} and g_{ij} we find

$$\sum_{n=1}^{3} \Gamma^n_{ij} g_{nm} = \frac{1}{2} \left(\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij} \right). \tag{50}$$

Multiplying by the inverse metric element g^{mk}, summing over m, and using $\sum_{m=1}^{3} g_{nm} g^{mk} = \delta^k_n$ we find

$$\Gamma^k_{ij} = \sum_{m=1}^{3} \frac{1}{2} g^{mk} \left(\partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij} \right). \tag{51}$$

Readers familiar with General Relativity will immediately recall that this is the expression for Christoffel's symbols.
We need an expression for $\sum_{i=1}^{3} \Gamma^i_{ij}$ which is

$$\sum_{i=1}^{3} \Gamma^i_{ij} = \sum_{i,k=1}^{3} \frac{1}{2} g^{ik} \left(\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij} \right), \tag{52}$$

$$= \sum_{i,k=1}^{3} \frac{1}{2} g^{ik} \partial_j g_{ik}, \tag{53}$$

where we have used the fact reshuffling indices cancels the first and third term, and the middle term survives. To evaluate this, we recall from Linear Algebra that for a non-singular matrix M, we have the following identity

$$\ln\left[\det(M)\right] = \mathrm{tr}\left(\ln M\right). \tag{54}$$

Using this one should be able to prove the important relation $\frac{\partial_j[\det(M)]}{\det(M)} = \mathrm{tr}\left(M^{-1}\partial_j M\right)$. Applying this for metric g_{ij}, we should get $\frac{\partial_j g}{g} = \sum_{i,k=1}^{3} g^{ik} \partial_j g_{ik}$, and consequently $\sum_{i=1}^{3} \Gamma^i_{ij} = \frac{\partial_j \sqrt{g}}{\sqrt{g}}$, where $g = \det(g_{ij})$.
Let us use this in Eq. (37) to get

$$\vec{\nabla} \cdot \vec{A} = \sum_{i=1}^{3} \partial_i \left(\frac{A^i}{h_i} \right) + \sum_{i,j=1}^{3} \frac{A^j}{h_j} \Gamma^i_{ij}, \tag{55}$$

$$= \sum_{i=1}^{3} \partial_i \left(\frac{A^i}{h_i} \right) + \sum_{j=1}^{3} \frac{A^j}{h_j} \frac{\partial_j \sqrt{g}}{\sqrt{g}}, \tag{56}$$

$$= \sum_{i=1}^{3} \left[\partial_i \left(\frac{A^i}{h_i} \right) + \frac{A^i}{h_i} \frac{\partial_i \sqrt{g}}{\sqrt{g}} \right], \tag{57}$$

$$= \frac{1}{\sqrt{g}} \sum_{i=1}^{3} \partial_i \left(\frac{\sqrt{g} A^i}{h_i} \right). \tag{58}$$

This formula is general for divergence of a vector. For metric (20) we have $\sqrt{g} = h_1 h_2 h_3$, and the above formula simplifies

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q^1} \left(A^1 h_2 h_3 \right) + \frac{\partial}{\partial q^2} \left(A^2 h_3 h_1 \right) + \frac{\partial}{\partial q^3} \left(A^3 h_1 h_2 \right) \right]. \tag{59}$$

3.3 Curl

In this section we will obtain a general formula for curl in curvilinear coordinates. To facilitate the calculation, we observe the following

$$\vec{\nabla} q^i = \sum_{j=1}^{3} \frac{\hat{q}_j}{h_j} \partial_j q^i = \frac{\hat{q}_i}{h_i}. \tag{60}$$

Then any vector can be written as

$$\vec{A} = \sum_{i=1}^{3} A^i \hat{q}_i = \sum_{i=1}^{3} A^i h_i \vec{\nabla} q^i. \tag{61}$$

Now we apply curl to both sides to get[5]

$$\vec{\nabla} \times \vec{A} = \sum_{i=1}^{3} \left[A^i h_i \vec{\nabla} \times \vec{\nabla} q^i - \vec{\nabla} q^i \times \vec{\nabla}(A^i h_i) \right]. \tag{62}$$

Since $\vec{\nabla} \times \vec{\nabla} \vec{V} = 0$ for any vector, in the above equation only the second term survives. Then we perform the following calculations

$$\vec{\nabla} \times \vec{A} = -\sum_{i=1}^{3} \vec{\nabla} q^i \times \vec{\nabla}(A^i h_i), \tag{63}$$

$$= -\sum_{i,j=1}^{3} \frac{1}{h_i h_j} \partial_j (A^i h_i) \hat{q}_i \times \hat{q}_j, \tag{64}$$

$$= -\sum_{i,j,k=1,3} \frac{\partial_j (A^i h_i)}{h_i h_j} \epsilon_{ijk} \hat{q}_k. \tag{65}$$

where ϵ_{ijk} is the totally antisymmetric symbol with the following definition

$$\epsilon_{ijk} = \begin{cases} 1 & \text{when } (i,j,k) = (1,2,3) \text{ or any even permutation,} \\ -1 & \text{when } (i,j,k) \text{ is an odd permutation of } (1,2,3). \end{cases} \tag{66}$$

With a reshuffling of indices, we can write

$$\vec{\nabla} \times \vec{A} = \sum_{i,j,k=1}^{3} \frac{\partial_i (A^i h_j)}{h_i h_j} \epsilon_{ijk} \hat{q}_k = \sum_{i,j,k=1}^{3} \frac{\partial_i (A^i h_j)}{h_i h_j h_k} \epsilon_{ijk} \vec{q}_k. \tag{67}$$

This can also be written as

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{q}_1 & h_2 \hat{q}_2 & h_3 \hat{q}_3 \\ \frac{\partial}{\partial q^1} & \frac{\partial}{\partial q^2} & \frac{\partial}{\partial q^3} \\ A^1 h_1 & A^2 h_2 & A^3 h_3 \end{vmatrix}. \tag{68}$$

[5] We use the following vector identity $\vec{\nabla} \times (f\vec{V}) = f\vec{\nabla} \times \vec{V} - \vec{V} \times \vec{\nabla} f$ for a scalar f and a vector $\vec{V}$.

3.4 Laplacian

After having the expressions of gradient and divergence, it is easy to find an expression for Laplacian valid for a general curvilinear coordinate system. Recall that the Laplacian is defined by

$$\nabla^2\Phi = \vec{\nabla}\cdot\vec{\nabla}\Phi. \tag{69}$$

Using the expression of Eq. (29), we can write

$$\vec{\nabla}\Phi = \frac{\hat{q}_1}{h_1}\frac{\partial\Phi}{\partial q^1} + \frac{\hat{q}_2}{h_2}\frac{\partial\Phi}{\partial q^2} + \frac{\hat{q}_3}{h_3}\frac{\partial\Phi}{\partial q^3}. \tag{70}$$

Then using the expression (59) we find the expression of Laplacian

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial q^1}\right) + \frac{\partial}{\partial q^2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial q^2}\right) + \frac{\partial}{\partial q^3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial q^3}\right)\right]. \tag{71}$$

4 Summary

In these notes we have obtained the expressions of vector operators, namely gradient, divergence, curl, and Laplacian in any curvilinear coordinate system. For a quick reference, we summarize here all the necessary formulae.

The differential distance vector:

$$\vec{d\ell} = h_1dq^1\hat{q}_1 + h_2dq^2\hat{q}_2 + h_3dq^3\hat{q}_3. \tag{72}$$

The metric:

$$ds^2 = (h_1)^2(dq^1)^2 + (h_2)^2(dq^2)^2 + (h_3)^2(dq^3)^2. \tag{73}$$

Representation of a vector:

$$\vec{A} = A^1\hat{q}_1 + A^2\hat{q}_2 + A^3\hat{q}_3. \tag{74}$$

The gradient operator:

$$\vec{\nabla} = \frac{\hat{q}_1}{h_1}\frac{\partial}{\partial q^1} + \frac{\hat{q}_2}{h_2}\frac{\partial}{\partial q^2} + \frac{\hat{q}_3}{h_3}\frac{\partial}{\partial q^3}. \tag{75}$$

The divergence of a vector:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q^1}\left(A^1h_2h_3\right) + \frac{\partial}{\partial q^2}\left(A^2h_3h_1\right) + \frac{\partial}{\partial q^3}\left(A^3h_1h_2\right)\right]. \tag{76}$$

The curl of a vector

$$\vec{\nabla}\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{q}_1 & h_2\hat{q}_2 & h_3\hat{q}_3 \\ \frac{\partial}{\partial q^1} & \frac{\partial}{\partial q^2} & \frac{\partial}{\partial q^3} \\ A^1h_1 & A^2h_2 & A^3h_3 \end{vmatrix}. \tag{77}$$

The Laplacian of a scalar:

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial q^1}\right) + \frac{\partial}{\partial q^2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial q^2}\right) + \frac{\partial}{\partial q^3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial q^3}\right)\right]. \tag{78}$$

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